

# A Liouville theorem for $\alpha$ -harmonic functions in $\mathbb{R}_+^n$

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## Abstract

In this paper, we consider  $\alpha$ -harmonic functions in the half space  $\mathbb{R}_+^n$ :

$$\begin{cases} (-\Delta)^{\alpha/2} u(x) = 0, & u(x) > 0, & x \in \mathbb{R}_+^n, \\ u(x) \equiv 0, & & x \notin \mathbb{R}_+^n. \end{cases} \quad (1)$$

We prove that all solutions of (1) have to assume the form

$$u(x) = \begin{cases} Cx_n^{\alpha/2}, & x \in \mathbb{R}_+^n, \\ 0, & x \notin \mathbb{R}_+^n, \end{cases} \quad (2)$$

for some positive constant  $C$ .

**Key words** The fractional Laplacian,  $\alpha$ -harmonic functions, uniqueness of solutions, Liouville theorem, Poisson representation.

## 1 Introduction

The fractional Laplacian in  $\mathbb{R}^n$  is a nonlocal pseudo-differential operator, assuming the form

$$(-\Delta)^{\alpha/2} u(x) = C_{n,\alpha} \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^n \setminus B_\epsilon(x)} \frac{u(x) - u(z)}{|x - z|^{n+\alpha}} dz, \quad (3)$$

where  $\alpha$  is any real number between 0 and 2. This operator is well defined in  $\mathcal{S}$ , the Schwartz space of rapidly decreasing  $C^\infty$  functions in  $\mathbb{R}^n$ . In this space, it can also be equivalently defined in terms of the Fourier transform

$$(-\Delta)^{\alpha/2}u(\xi) = |\xi|^\alpha \hat{u}(\xi),$$

where  $\hat{u}$  is the Fourier transform of  $u$ . One can extend this operator to a wider space of distributions.

Let

$$L_\alpha = \{u : \mathbb{R}^n \rightarrow \mathbb{R} \mid \int_{\mathbb{R}^n} \frac{|u(x)|}{1 + |x|^{n+\alpha}} dx < \infty\}.$$

Then in this space, we defined  $(-\Delta)^{\alpha/2}u$  as a distribution by

$$\langle (-\Delta)^{\alpha/2}u(x), \phi \rangle = \int_{\mathbb{R}^n} u(x)(-\Delta)^{\alpha/2}\phi(x)dx, \quad \forall \phi \in C_0^\infty(\mathbb{R}^n).$$

Let

$$\mathbb{R}_+^n = \{x = (x_1, \dots, x_n) \mid x_n > 0\}$$

be the upper half space. We say that  $u$  is  $\alpha$ -harmonic in the upper half space if

$$\int_{\mathbb{R}^n} u(x)(-\Delta)^{\alpha/2}\phi(x)dx = 0, \quad \forall \phi \in C_0^\infty(\mathbb{R}_+^n).$$

In this paper, we consider the Dirichlet problem for  $\alpha$ -harmonic functions

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = 0, & u(x) > 0, & x \in \mathbb{R}_+^n, \\ u(x) \equiv 0, & & x \notin \mathbb{R}_+^n, \end{cases} \quad (4)$$

It is well-known that

$$u(x) = \begin{cases} Cx_n^{\alpha/2}, & x \in \mathbb{R}_+^n, \\ 0, & x \notin \mathbb{R}_+^n, \end{cases}$$

is a family of solutions for problem (4) with any positive constant  $C$ .

A natural question is: *Are there any other solutions?*

Our main objective here is to answer this question and prove

**Theorem 1** *Let  $0 < \alpha < 2$ ,  $u \in L_\alpha$ . Assume  $u$  is a solution of*

$$\begin{cases} (-\Delta)^{\alpha/2}u(x) = 0, & u(x) > 0, & x \in \mathbb{R}_+^n, \\ u(x) \equiv 0, & & x \notin \mathbb{R}_+^n. \end{cases} \quad (5)$$

then

$$u(x) = \begin{cases} Cx_n^{\alpha/2}, & x \in \mathbb{R}_+^n, \\ 0, & x \notin \mathbb{R}_+^n, \end{cases} \quad (6)$$

for some positive constant  $C$ .

We will prove this theorem in the next section.

## 2 The Proof of the Liouville Theorem

In this section, we prove Theorem 1. The main ideas are the following.

We first obtain the Poisson representation of the solutions. We show that for  $|x - x_r| < r$

$$u(x) = \int_{|y-x_r|>r} P_r(x - x_r, y - x_r) u(y) dy, \quad (7)$$

where  $x_r = (0, \dots, 0, r)$ , and  $P_r(x - x_r, y - x_r)$  is the Poisson kernel for  $|x - x_r| < r$ :

$$\begin{aligned} & P_r(x - x_r, y - x_r) \\ = & \begin{cases} \frac{\Gamma(n/2)}{\pi^{\frac{n}{2}+1}} \sin \frac{\pi\alpha}{2} \left[ \frac{r^2 - |x - x_r|^2}{|y - x_r|^2 - r^2} \right]^{\frac{\alpha}{2}} \frac{1}{|x - y|^n}, & |y - x_r| > r, \\ 0, & \text{elsewhere.} \end{cases} \end{aligned}$$

Then, for each fixed  $x \in \mathbb{R}_+^n$ , we evaluate first derivatives of  $u$  by using (7). Letting  $r \rightarrow \infty$ , we derive

$$\frac{\partial u}{\partial x_i}(x) = 0, \quad i = 1, 2, \dots, n-1.$$

and

$$\frac{\partial u}{\partial x_n}(x) = \frac{\alpha}{2x_n} u(x).$$

These yield the desired results.

In the following, we use  $C$  to denote various positive constants.

*Step 1.*

In this step, we obtain the Poisson representation (7) for the solutions of (4).

Let

$$\hat{u}(x) = \begin{cases} \int_{|y-x_r|>r} P_r(x-x_r, y-x_r)u(y)dy, & |x-x_r| < r, \\ u(x), & |x-x_r| \geq r. \end{cases} \quad (8)$$

We will prove that  $\hat{u}$  is  $\alpha$ -harmonic in  $B_r(x_r)$ . The proof is similar to that in [CL]. It is quite long and complex, hence for reader's convenience, we will present it in the next section.

Let  $w(x) = u - \hat{u}$ , then

$$\begin{cases} (-\Delta)^{\alpha/2}w(x) = 0, & |x-x_r| < r, \\ w(x) \equiv 0, & |x-x_r| \geq r. \end{cases} \quad (9)$$

To show that  $w \equiv 0$ , we employ the following Maximum Principle.

**Lemma 2.1** (Silvestre, [Si]) *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^n$ , and assume that  $v$  is a lower semi-continuous function on  $\overline{\Omega}$  satisfying*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}}v \geq 0, & \text{in } \Omega, \\ v \geq 0, & \text{on } \mathbb{R}^n \setminus \Omega. \end{cases} \quad (10)$$

*then  $v \geq 0$  in  $\Omega$ .*

Applying this lemma to both  $v = w$  and  $v = -w$ , we conclude that

$$w(x) \equiv 0.$$

Hence

$$\hat{u}(x) \equiv u(x).$$

This verifies (7).

*Step 2.*

We will show that for each fixed  $x \in \mathbb{R}_+^n$ ,

$$\frac{\partial u}{\partial x_i}(x) = 0, \quad i = 1, 2, \dots, n-1. \quad (11)$$

and

$$\frac{\partial u}{\partial x_n}(x) = \frac{\alpha}{2x_n}u(x). \quad (12)$$

From (11), we conclude that  $u(x) = u(x_n)$ , and this, together with (12), immediately implies

$$u(x) = Cx_n^{\alpha/2}, \quad (13)$$

therefore

$$u(x) = \begin{cases} Cx_n^{\alpha/2}, & x \in \mathbb{R}_+^n, \\ 0, & x \notin \mathbb{R}_+^n. \end{cases} \quad (14)$$

And this is what we want to derive.

Now, what left is to prove (11) and (12). Through an elementary calculation, one can derive that, for  $i = 1, 2, \dots, n-1$ ,

$$\begin{aligned} \frac{\partial u}{\partial x_i}(x) &= \int_{|y-x_r|>r} \left( \frac{-\alpha x_i}{r^2 - |x-x_r|^2} + \frac{n(y_i - x_i)}{|y-x|^2} \right) P_r(x-x_r, y-x_r) u(y) dy \\ &= \int_{|y-x_r|>r} \frac{-\alpha x_i}{r^2 - |x-x_r|^2} P_r(x-x_r, y-x_r) u(y) dy \\ &\quad + \int_{|y-x_r|>r} \frac{n(y_i - x_i)}{|y-x|^2} P_r(x-x_r, y-x_r) u(y) dy \\ &:= I_1 + I_2. \end{aligned} \quad (15)$$

For each fixed  $x \in B_r(x_r) \subset \mathbb{R}_+^n$  and for any given  $\epsilon > 0$ , we have

$$\begin{aligned} |I_1| &= \left| \int_{|y-x_r|>r} \frac{-\alpha x_i}{r^2 - |x-x_r|^2} P_r(x-x_r, y-x_r) u(y) dy \right| \\ &\leq \int_{|y-x_r|>r} \left| \frac{-\alpha x_i}{r^2 - |x-x_r|^2} \right| P_r(x-x_r, y-x_r) u(y) dy \\ &= \left| \frac{\alpha x_i}{r^2 - |x-x_r|^2} \right| \int_{|y-x_r|>r} P_r(x-x_r, y-x_r) u(y) dy \\ &= \left| \frac{\alpha x_i}{r^2 - |x-x_r|^2} \right| u(x) \\ &\leq \frac{C}{r} \\ &< \epsilon, \quad \text{for sufficiently large } r. \end{aligned} \quad (16)$$

Here and in below, the letter  $C$  stands for various positive constants.

It is more delicate to estimate  $I_2$ . For each  $R > 0$ , we divide the region  $|y-x_r| > r$  into two parts: one inside the ball  $|y| < R$  and one outside the ball.

$$\begin{aligned}
|I_2| &= \left| \int_{\substack{|y-x_r|>r \\ |y|>R}} \frac{n(y_i - x_i)}{|y - x|^2} P_r(x - x_r, y - x_r) u(y) dy \right| \\
&\leq \int_{|y-x_r|>r} \left| \frac{n(y_i - x_i)}{|y - x|^2} \right| P_r(x - x_r, y - x_r) u(y) dy \\
&= \int_{\substack{|y-x_r|>r \\ |y|>R}} \left| \frac{n(y_i - x_i)}{|y - x|^2} \right| P_r(x - x_r, y - x_r) u(y) dy \\
&\quad + \int_{\substack{|y-x_r|>r \\ |y|\leq R}} \left| \frac{n(y_i - x_i)}{|y - x|^2} \right| P_r(x - x_r, y - x_r) u(y) dy \\
&:= I_{21} + I_{22}.
\end{aligned} \tag{17}$$

For the  $\epsilon > 0$  given above, when  $|y| > R$ , we can easily derive

$$\left| \frac{n(y_i - x_i)}{|y - x|^2} \right| \leq \frac{n}{|y - x|} \leq \frac{C}{R} < \epsilon,$$

for sufficiently large  $R$ . Fix this  $R$ , then

$$\begin{aligned}
I_{21} &< \int_{\substack{|y-x_r|>r \\ |y|>R}} \epsilon P_r(x - x_r, y - x_r) u(y) dy \\
&\leq \epsilon \int_{|y-x_r|>r} P_r(x - x_r, y - x_r) u(y) dy \\
&= \epsilon u(x) \\
&\leq C\epsilon.
\end{aligned} \tag{18}$$

To estimate  $I_{22}$ , we employ the expression of the Poisson kernel.

$$\begin{aligned}
I_{22} &= C \int_{\substack{|y-x_r|>r \\ |y|\leq R}} \left[ \frac{r^2 - |x - x_r|^2}{|y - x_r|^2 - r^2} \right]^{\alpha/2} \frac{u(y)}{|x - y|^n} \left| \frac{n(y_i - x_i)}{|y - x|^2} \right| dy \\
&\leq C \int_{\substack{|y-x_r|>r \\ |y|\leq R}} \left[ \frac{2x_nr - |x|^2}{2y_nr - |y|^2} \right]^{\alpha/2} \frac{u(y)}{|x - y|^n} \frac{1}{|y - x|} dy \\
&\leq C \int_{\substack{|y-x_r|>r \\ |y|\leq R}} \left[ \frac{2x_nr - |x|^2}{2y_nr - |y|^2} \right]^{\alpha/2} \frac{u(y)}{|x - y|^{n+1}} dy \\
&= C \int_{\substack{|y-x_r|>r \\ |y|\leq R, y_n>0}} \left[ \frac{2x_nr - |x|^2}{2y_nr - |y|^2} \right]^{\alpha/2} \frac{u(y)}{|x - y|^{n+1}} dy \\
&\leq C_R \int_{\substack{|y-x_r|>r \\ |y|\leq R, y_n>0}} \left[ \frac{2x_nr - |x|^2}{2y_nr - |y|^2} \right]^{\alpha/2} \frac{1}{|x - y|^{n+1}} dy. \tag{19}
\end{aligned}$$

Here we have used the fact that the  $\alpha$ -harmonic function  $u$  is bounded in the region

$$D_{R,r} = \{y = (y', y_n) \mid |y - x_r| > r, |y| < R, y_n > 0\}.$$

The bound depends on  $R$ , however is independent of  $r$ , since  $D_{R,r_1} \subset D_{R,r_2}$  for  $r_1 > r_2$ . For each such fixed open domain  $D_{R,r}$ , the bound of the  $\alpha$ -harmonic function  $u$  can be derived from the interior smoothness ( see, for instance [BKN] and [FW]) and the estimate up to the boundary ( see [RS]).

Set  $y = (y', y_n)$ ,  $\sigma = |y'|$ , for fixed  $x$  and sufficiently large  $r$ , we have

$$\begin{aligned}
\left[ \frac{2x_nr - |x|^2}{2y_nr - |y|^2} \right]^{\alpha/2} &\leq \frac{Cr^{\alpha/2}}{|2y_nr - |y|^2|^{\alpha/2}} \\
&= \frac{Cr^{\alpha/2}}{|\sigma^2 - 2y_nr + y_n^2|^{\alpha/2}} \\
&= \frac{Cr^{\alpha/2}}{|(y_n - r)^2 + \sigma^2 - r^2|^{\alpha/2}}. \tag{20}
\end{aligned}$$

and

$$\frac{1}{|x - y|^{n+1}} \leq \frac{C}{(1 + |y|)^{n+1}} \leq \frac{C}{(1 + |y'|)^{n+1}} = \frac{C}{(1 + \sigma)^{n+1}}. \tag{21}$$

For convenience of estimate, we amplify the domain  $D_{R,r}$  a little bit. Define

$$\hat{D}_{R,r} = \{y = (y', y_n) \in \mathbb{R}_+^n \mid |y - x_r| > r, |y'| \leq R, 0 < y_n < \bar{y}_n\}.$$

Here  $\bar{y}_n$  satisfies

$$(\bar{y}_n - r)^2 + \sigma^2 - r^2 = 0, \quad (22)$$

so that  $\bar{y} = (y', \bar{y}_n) \in \partial \hat{D}_{R,r} \cap \partial B_r(x_r)$ . Then it is easy to see that

$$D_{R,r} \setminus \hat{D}_{R,r} \quad (23)$$

From (22), for sufficiently large  $r$  (much larger than  $R$ ), we have

$$\bar{y}_n = r - \sqrt{r^2 - \sigma^2}.$$

Set

$$y_n = r - s\sqrt{r^2 - \sigma^2}. \quad (24)$$

Then for  $0 < y_n < \bar{y}_n$ ,

$$1 < s < \frac{r}{\sqrt{r^2 - \sigma^2}}, \quad (25)$$

and

$$dy_n = -\sqrt{r^2 - \sigma^2} ds. \quad (26)$$

Continuing from the right side of (19), we integrate in the direction of  $y_n$  first, and then integrate with respect to  $y'$ , setting  $r$  sufficiently large



(much larger than  $R$ ), by(19), (20), (21), (23),( 24), (25), and(26), we derive

$$\begin{aligned}
I_{22} &\leq C \int_{\hat{D}_{R,r}} \left[ \frac{2x_n r - |x|^2}{2y_n r - |y|^2} \right]^{\alpha/2} \frac{1}{|x-y|^{n+1}} dy \\
&\leq C \int_{|y'| < R} \int_0^{\bar{y}_n} \frac{r^{\alpha/2}}{|(y_n - r)^2 + \sigma^2 - r^2|^{\alpha/2}} dy_n \frac{1}{(1+\sigma)^{n+1}} dy' \\
&\leq C \int_0^R \int_0^{\bar{y}_n} \frac{r^{\alpha/2}}{|(y_n - r)^2 + \sigma^2 - r^2|^{\alpha/2}} dy_n \frac{1}{(1+\sigma)^{n+1}} \sigma^{n-2} d\sigma \tag{27}
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^R \int_1^{\frac{r}{\sqrt{r^2 - \sigma^2}}} \frac{r^{\alpha/2}}{[(s\sqrt{r^2 - \sigma^2})^2 - (r^2 - \sigma^2)]^{\alpha/2}} \left| \sqrt{r^2 - \sigma^2} \right| ds \frac{\sigma^{n-2}}{(1+\sigma)^{n+1}} d\sigma \\
&= C \int_0^R r^{\alpha/2} (r^2 - \sigma^2)^{\frac{1-\alpha}{2}} \int_1^{\frac{r}{\sqrt{r^2 - \sigma^2}}} \frac{1}{(s^2 - 1)^{\alpha/2}} ds \frac{\sigma^{n-2}}{(1+\sigma)^{n+1}} d\sigma \\
&\leq C \int_0^R r^{\alpha/2} (r^2 - \sigma^2)^{\frac{1-\alpha}{2}} \int_1^{\frac{r}{\sqrt{r^2 - \sigma^2}}} \frac{1}{(s-1)^{\alpha/2}} ds \frac{\sigma^{n-2}}{(1+\sigma)^{n+1}} d\sigma \tag{28}
\end{aligned}$$

$$\begin{aligned}
&\leq C \int_0^R r^{\alpha/2} r^{1-\alpha} \int_1^{\frac{r}{\sqrt{r^2 - \sigma^2}}} \frac{1}{(s-1)^{\alpha/2}} ds \frac{\sigma^{n-2}}{(1+\sigma)^{n+1}} d\sigma \tag{29} \\
&= C \int_0^R r^{1-\alpha/2} \left( \frac{r}{\sqrt{r^2 - \sigma^2}} - 1 \right)^{1-\alpha/2} \frac{\sigma^{n-2}}{(1+\sigma)^{n+1}} d\sigma
\end{aligned}$$

$$\begin{aligned}
&= C \int_0^R r^{1-\alpha/2} \left( \frac{\sigma^2}{(r + \sqrt{r^2 - \sigma^2}) \sqrt{r^2 - \sigma^2}} \right)^{1-\alpha/2} \frac{\sigma^{n-2}}{(1+\sigma)^{n+1}} d\sigma \\
&\leq C \int_0^R r^{1-\alpha/2} \left( \frac{1}{r^2} \right)^{1-\alpha/2} \frac{\sigma^{n-2+2-\alpha}}{(1+\sigma)^{n+1}} d\sigma \\
&= C r^{\alpha/2-1} \int_0^R \frac{\sigma^{n-\alpha}}{(1+\sigma)^{n+1}} d\sigma \\
&\leq C r^{\alpha/2-1} \\
&= \frac{C}{r^{1-\alpha/2}}. \tag{30}
\end{aligned}$$

In the above, we derived (27) by letting  $|y'| = \sigma$ . (28) is valid because

$$\begin{aligned}
\frac{1}{(s^2 - 1)^{\alpha/2}} &= \frac{1}{(s + 1)^{\alpha/2}} \frac{1}{(s - 1)^{\alpha/2}} \\
&\leq \frac{1}{(1 + 1)^{\alpha/2}} \frac{1}{(s - 1)^{\alpha/2}} \\
&= \frac{1}{2^{\alpha/2}} \frac{1}{(s - 1)^{\alpha/2}} \\
&\leq \frac{1}{(s - 1)^{\alpha/2}}.
\end{aligned}$$

Since  $R$  is fixed and  $\sigma^2 \leq R^2$ , when  $r$  is sufficiently large (much larger than  $R$ ), we have  $r^2 - \sigma^2 > 0$ , and the value of  $(r^2 - \sigma^2)^{\frac{1-\alpha}{2}}$  can be dominated by  $(r^2)^{\frac{1-\alpha}{2}}$  (i.e.  $r^{1-\alpha}$ ), this verifies (29).

For the  $\epsilon > 0$  given above and the fixed  $R$ , since  $0 < \alpha < 2$ , then by (30) we can easily get

$$I_{22} \leq C \frac{1}{r^{1-\alpha/2}} < \epsilon, \quad (31)$$

for sufficiently large  $r$ .

From (15), (16), (17), (18), and (31), we derive

$$\left| \frac{\partial u}{\partial x_i}(x) \right| < C\epsilon, \quad (32)$$

for sufficiently large  $R$  and much larger  $r$ .

The fact that  $\epsilon$  is arbitrary implies

$$\left| \frac{\partial u}{\partial x_i}(x) \right| = 0. \quad (33)$$

This proves (11).

Now, let's prove (12). Similarly, for fixed  $x \in B_r(x_r) \subset \mathbb{R}_+^n$ , through an elementary calculation, one can derive that

$$\begin{aligned}
\frac{\partial u}{\partial x_n}(x) &= \int_{|y-x_r|>r} \left( \frac{\alpha(r-x_n)}{r^2 - |x-x_r|^2} + \frac{n(y_n-x_n)}{|y-x|^2} \right) P_r(x-x_r, y-x_r) u(y) dy \\
&= \int_{|y-x_r|>r} \frac{\alpha(r-x_n)}{r^2 - |x-x_r|^2} P_r(x-x_r, y-x_r) u(y) dy \\
&\quad + \int_{|y-x_r|>r} \frac{n(y_n-x_n)}{|y-x|^2} P_r(x-x_r, y-x_r) u(y) dy \\
&:= J_1 + J_2.
\end{aligned} \quad (34)$$

Similarly to  $I_2$ , for sufficiently large  $r$ , we can also derive

$$|J_2| \leq C\epsilon, \quad (35)$$

for any  $\epsilon > 0$ . That is

$$J_2 \rightarrow 0, \quad \text{as } r \rightarrow \infty. \quad (36)$$

Now we estimate  $J_1$ .

$$\begin{aligned} J_1 &= \frac{\alpha(r - x_n)}{r^2 - |x - x_r|^2} \int_{|y - x_r| > r} P_r(x - x_r, y - x_r) u(y) dy \\ &= \frac{\alpha(r - x_n)}{2x_n r - |x|^2} u(x). \end{aligned}$$

It follows that

$$J_1 \rightarrow \frac{\alpha}{2x_n} u(x), \quad \text{as } r \rightarrow \infty. \quad (37)$$

By (34), (36), and (37), for each fixed  $x \in B_r(x_r) \subset \mathbb{R}_+^n$ , letting  $r \rightarrow \infty$ , we arrive at

$$\frac{\partial u}{\partial x_n}(x) = \frac{\alpha}{2x_n} u(x).$$

This verifies (12), and hence completes the proof of Theorem 1.

### 3 $\hat{u}(x)$ is $\alpha$ -harmonic in $B_r(x_r)$

In this section, we prove

**Theorem 3.1**  *$\hat{u}(x)$  defined by (8) in the previous section is  $\alpha$ -harmonic in  $B_r(x_r)$ .*

The proof consists of two parts. First we show that  $\hat{u}$  is harmonic in the average sense (Lemma 3.1), then we show that it is  $\alpha$ -harmonic (Lemma 3.2).

Let

$$\varepsilon_\alpha^{(r)}(x) = \begin{cases} 0, & |x| < r. \\ \frac{\Gamma(n/2)}{\pi^{\frac{n}{2}+1}} \sin \frac{\pi\alpha}{2} \frac{r^\alpha}{(|x|^2 - r^2)^{\frac{\alpha}{2}} |x|^n}, & |x| > r. \end{cases} \quad (38)$$

We say that  $u$  is  $\alpha$ -harmonic in the average sense (see [L]) if for small  $r$ ,

$$\varepsilon_\alpha^{(r)} * u(x) = u(x).$$

Let

$$P_r(x - x_r, y - x_r) = \begin{cases} \frac{\Gamma(n/2)}{\pi^{\frac{n}{2}+1}} \sin \frac{\pi\alpha}{2} \left[ \frac{r^2 - |x - x_r|^2}{|y - x_r|^2 - r^2} \right]^{\frac{\alpha}{2}} \frac{1}{|x - y|^n}, & |y - x_r| > r, |x - x_r| < r \\ 0, & \text{elsewhere.} \end{cases} \quad (39)$$

**Lemma 3.1** *Let  $u(x)$  be any measurable function outside  $B_r(x_r)$  for which*

$$\int_{\mathbb{R}^n} \frac{|u(z)|}{(1 + |z - x_r|)^{n+\alpha}} dz < \infty. \quad (40)$$

*Let*

$$\hat{u}(x) = \begin{cases} \int_{|y-x_r|>r} P_r(y - x_r, x - x_r) u(y) dy, & |x - x_r| < r, \\ u(x), & |x - x_r| \geq r. \end{cases} \quad (41)$$

*Then  $\hat{u}(x)$  is  $\alpha$ -harmonic in the average sense in  $B_r(x_r)$ , i.e. for sufficiently small  $\delta$ , we have*

$$(\varepsilon_\alpha^{(\delta)} * \hat{u})(x) = \hat{u}(x), \quad |x - x_r| < r, \quad (42)$$

*where  $*$  is the convolution.*

**Proof.**

The outline is as follows.

i) Approximate  $u$  by a sequence of smooth, compactly supported functions  $\{u_k\}$ , such that  $u_k(x) \rightarrow u(x)$  and

$$\int_{|z-x_r|>r} \frac{|u_k(z) - u(z)|}{|z - x_r|^n (|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} dz \rightarrow 0. \quad (43)$$

This is possible under our assumption (40).

ii) For each  $u_k$ , find a signed measure  $\nu_k$  such that  $\text{supp } \nu_k \subset B_r^c(x_r)$  and

$$u_k(x) = U_\alpha^{\nu_k}(x), \quad |x - x_r| > r.$$

Then

$$\hat{u}_k(x) = U_\alpha^{\nu_k}(x), \quad |x - x_r| < r.$$

iii) It is easy to see that  $\hat{u}_k(x)$  is  $\alpha$ -harmonic in the average sense for  $|x - x_r| < r$ . That is, for each fixed small  $\delta > 0$ ,

$$(\varepsilon_\alpha^{(\delta)} * \hat{u}_k)(x) = \hat{u}_k(x). \quad (44)$$

By showing that as  $k \rightarrow \infty$

$$\varepsilon_\alpha^{(\delta)} * \hat{u}_k \rightarrow \varepsilon_\alpha^{(\delta)} * \hat{u},$$

and

$$\hat{u}_k \rightarrow \hat{u},$$

we arrive at

$$(\varepsilon_\alpha^{(\delta)} * \hat{u})(x) = \hat{u}(x), \quad |x - x_r| < r.$$

Now we carry out the details.

i) There are several ways to construct such a sequence  $\{u_k\}$ . One is to use the mollifier. Let

$$u|_{B_k(x_r)}(x) = \begin{cases} u(x), & |x - x_r| < k, \\ 0, & |x - x_r| \geq k, \end{cases} \quad (45)$$

and

$$J_\epsilon(u|_{B_k(x_r)})(x) = \int_{\mathbb{R}^n} j_\epsilon(x - y) u|_{B_k(x_r)}(y) dy. \quad (46)$$

For any  $\delta > 0$ , let  $k$  be sufficiently large (larger than  $r$ ) such that

$$\int_{|z - x_r| \geq k} \frac{|u(z)|}{|z - x_r|^n (|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} dz < \frac{\delta}{2}. \quad (47)$$

For each such  $k$ , choose  $\epsilon_k$  such that

$$\int_{B_{k+1} \setminus B_r} \frac{|u_k(z) - u|_{B_k(x_r)}(z)|}{|z - x_r|^n (|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} dz < \frac{\delta}{2}, \quad (48)$$

where  $u_k = J_{\epsilon_k}(u|_{B_k(x_r)})$ . It then follows that

$$\begin{aligned} & \int_{|z - x_r| > r} \frac{|u_k(z) - u(z)|}{|z - x_r|^n (|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} dz \\ & \leq \int_{B_{k+1}(x_r) \setminus B_r(x_r)} \frac{|u_k(z) - u|_{B_k(x_r)}(z)| + |u|_{B_k(x_r)}(z) - u(z)|}{|z - x_r|^n (|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} dz \\ & \quad + \int_{|z - x_r| > k+1} \frac{|u(z)|}{|z - x_r|^n (|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} dz \\ & = \int_{B_{k+1}(x_r) \setminus B_r(x_r)} \frac{|u_k(z) - u|_{B_k(x_r)}(z)|}{|z - x_r|^n (|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} dz + \int_{|z - x_r| \geq k} \frac{|u(z)|}{|z - x_r|^n (|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} dz \\ & < \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{aligned}$$

Therefore, as  $k \rightarrow \infty$ ,

$$\int_{|z-x_r|>r} \frac{|u_k(z) - u(z)|}{|z - x_r|^n (|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} dz \rightarrow 0. \quad (49)$$

ii) For each  $u_k$ , there exists a signed measure  $\psi_k$  such that

$$u_k(x) = U_\alpha^{\psi_k}(x). \quad (50)$$

Indeed, let  $\psi_k(x) = C(-\Delta)^{\frac{\alpha}{2}} u_k(x)$ , then

$$U_\alpha^{\psi_k}(x) = \int_{\mathbb{R}^n} \frac{C}{|x - y|^{n-\alpha}} (-\Delta)^{\frac{\alpha}{2}} u_k(y) dy \quad (51)$$

$$= \int_{\mathbb{R}^n} (-\Delta)^{\frac{\alpha}{2}} \left[ \frac{C}{|x - y|^{n-\alpha}} \right] u_k(y) dy \quad (52)$$

$$= \int_{\mathbb{R}^n} \delta(x - y) u_k(y) dy = u_k(x). \quad (53)$$

Here we have used the fact that  $\frac{C}{|x-y|^{n-\alpha}}$  is the fundamental solution of  $(-\Delta)^{\alpha/2}$ .

Let  $\psi_k|_{B_r(x_r)}$  be the restriction of  $\psi_k$  on  $B_r(x_r)$  and

$$\tilde{\psi}_k(y) = \int_{|x-x_r|<r} P_r(y - x_r, x - x_r) \psi_k|_{B_r(x_r)}(x) dx, \quad (54)$$

we have

$$U_\alpha^{\tilde{\psi}_k}(x) = U_\alpha^{\psi_k|_{B_r(x_r)}}(x), \quad |x - x_r| > r,$$

and  $\text{supp } \tilde{\psi}_k \subset B_r^c(x_r)$ . Here we use the fact (see (1.6.12') [L]) that

$$\frac{1}{|z - x|^{n-\alpha}} = \int_{|y-x_r|>r} \frac{P_r(y - x_r, x - x_r)}{|z - y|^{n-\alpha}} dy, \quad |x - x_r| < r, \quad |z - x_r| > r. \quad (55)$$

Let  $\nu_k = \psi_k - \psi_k|_{B_r(x_r)} + \tilde{\psi}_k$ , then  $\text{supp } \nu_k \subset B_r^c(x_r)$ , and

$$U_\alpha^{\nu_k}(x) = U_\alpha^{\psi_k}(x) + U_\alpha^{\tilde{\psi}_k}(x) - U_\alpha^{\psi_k|_{B_r(x_r)}}(x) = U_\alpha^{\psi_k}(x), \quad |x - x_r| > r.$$

That is

$$u_k(x) = U_\alpha^{\nu_k}(x), \quad |x - x_r| > r.$$

Again by (55), we deduce

$$\hat{u}_k(x) = U_\alpha^{\nu_k}(x), \quad |x - x_r| < r.$$

In this case  $\hat{u}_k$  is  $\alpha$ -harmonic (in the sense of average) in the region  $|x - x_r| < r$  (see [L]).

iii) For each fixed  $x$ , we first have

$$\hat{u}_k(x) \rightarrow \hat{u}(x).$$

In fact, by (49),

$$\begin{aligned} \hat{u}_k(x) - \hat{u}(x) &= \int_{|y-x_r|>r} P_r(y - x_r, x - x_r) [u_k(y) - u(y)] dy \\ &= C \int_{|y-x_r|>r} \frac{(r^2 - |x - x_r|^2)^{\frac{\alpha}{2}} [u_k(y) - u(y)]}{(|y - x_r|^2 - r^2)^{\frac{\alpha}{2}} |x - y|^n} dy \\ &\rightarrow 0. \end{aligned}$$

Next, we show that, for each fixed  $\delta > 0$  and fixed  $x$ ,

$$(\varepsilon_\alpha^{(\delta)} * \hat{u}_k)(x) \rightarrow (\varepsilon_\alpha^{(\delta)} * \hat{u})(x). \quad (56)$$

Indeed,

$$\begin{aligned} &(\varepsilon_\alpha^{(\delta)} * \hat{u}_k)(x) - (\varepsilon_\alpha^{(\delta)} * \hat{u})(x) \\ &= C \int_{|y-x|>\delta} \frac{\delta^\alpha [\hat{u}_k(y) - \hat{u}(y)]}{(|x - y|^2 - \delta^2)^{\frac{\alpha}{2}} |x - y|^n} dy \\ &= C \left\{ \int_{\substack{|y-x|>\delta \\ |y-x_r|<r-\eta}} \frac{\delta^\alpha [\hat{u}_k(y) - \hat{u}(y)]}{(|x - y|^2 - \delta^2)^{\frac{\alpha}{2}} |x - y|^n} dy \right. \\ &\quad + \int_{\substack{|y-x|>\delta \\ r-\eta<|y-x_r|<r}} \frac{\delta^\alpha [\hat{u}_k(y) - \hat{u}(y)]}{(|x - y|^2 - \delta^2)^{\frac{\alpha}{2}} |x - y|^n} dy \\ &\quad \left. + \int_{\substack{|y-x|>\delta \\ |y-x_r|>r}} \frac{\delta^\alpha [\hat{u}_k(y) - \hat{u}(y)]}{(|x - y|^2 - \delta^2)^{\frac{\alpha}{2}} |x - y|^n} dy \right\} \\ &= C(I_1 + I_2 + I_3). \end{aligned}$$

For each fixed  $x$  with  $|x - x_r| < r$ , choose  $\delta$  and  $\eta$  such that

$$B_\delta(x) \cap B_{r-2\eta}^c(x_r) = \emptyset.$$

It follows from (49) that as  $k \rightarrow \infty$ ,

$$I_3 = \int_{\substack{|y-x|>\delta \\ |y-x_r|>r}} \frac{\delta^\alpha [u_k(y) - u(y)]}{(|x-y|^2 - \delta^2)^{\frac{\alpha}{2}} |x-y|^n} dy \rightarrow 0. \quad (57)$$

$$\begin{aligned} I_2 &= \int_{\substack{|y-x|>\delta \\ r-\eta < |y-x_r| < r}} \frac{\delta^\alpha \int_{|z-x_r|>r} P_r(z-x_r, y-x_r) [u_k(z) - u(z)] dz}{(|x-y|^2 - \delta^2)^{\frac{\alpha}{2}} |x-y|^n} dy \\ &= C \delta^\alpha \int_{|z-x_r|>r} \frac{u_k(z) - u(z)}{(|z-x_r|^2 - r^2)^{\frac{\alpha}{2}}} \int_{\substack{|y-x|>\delta \\ r-\eta < |y-x_r| < r}} \frac{(r^2 - |y-x_r|^2)^{\frac{\alpha}{2}} dy}{(|x-y|^2 - \delta^2)^{\frac{\alpha}{2}} |x-y|^n |z-y|^n} dz \\ &= C \delta^\alpha \int_{|z-x_r|>r} \frac{u_k(z) - u(z)}{(|z-x_r|^2 - r^2)^{\frac{\alpha}{2}}} \cdot I_{21}(x, z) dz. \end{aligned}$$

Noting that in the ring  $r - \eta < |y - x_r| < r$ , we have

$$|x - y| > \eta + \delta.$$

It then follows that

$$\begin{aligned} &I_{21}(x, z) \\ &\leq \frac{1}{(2\eta\delta + \eta^2)^{\frac{\alpha}{2}} (\eta + \delta)^n} \int_{r-\eta < |y-x_r| < r} \frac{(r^2 - |y-x_r|^2)^{\frac{\alpha}{2}} dy}{|z-y|^n} \\ &= C \int_{r-\eta}^r (r^2 - \tau^2)^{\frac{\alpha}{2}} \left\{ \int_{S_\tau} \frac{1}{|z-y|^n} d\sigma_y \right\} d\tau \\ &= C \int_{r-\eta}^r (r^2 - \tau^2)^{\frac{\alpha}{2}} \left\{ \int_0^\pi \frac{\omega_{n-2}(\tau \sin \theta)^{n-2} \tau d\theta}{(\tau^2 + |z-x_r|^2 - 2\tau|z-x_r|\cos \theta)^{\frac{n}{2}}} \right\} d\tau \\ &= C \int_{r-\eta}^r (r^2 - \tau^2)^{\frac{\alpha}{2}} \frac{1}{\tau^n} \int_0^\pi \frac{\tau^{n-1} \sin^{n-2} \theta d\theta}{((\frac{|z-x_r|}{\tau})^2 - 2\frac{|z-x_r|}{\tau} \cos \theta + 1)^{\frac{n}{2}}} d\tau \quad (58) \end{aligned}$$

$$\begin{aligned} &= C \int_{r-\eta}^r \frac{(r^2 - \tau^2)^{\frac{\alpha}{2}}}{\tau} \frac{d\tau}{(\frac{|z-x_r|}{\tau})^{n-2} ((\frac{|z-x_r|}{\tau})^2 - 1)} \int_0^\pi \sin^{n-2} \theta d\theta \\ &< \frac{C r^{n-1}}{|z-x_r|^{n-2}} \int_{r-\eta}^r \frac{(r^2 - \tau^2)^{\frac{\alpha}{2}}}{|z-x_r|^2 - \tau^2} d\tau \\ &= \frac{C r^{n-1}}{|z-x_r|^{n-2}} \cdot J. \quad (59) \end{aligned}$$



In the above, to derive (59) from (58), we have made the following substitution (See Appendix in [L]):

$$\frac{\sin \theta}{\sqrt{\left(\frac{|z-x_r|}{\tau}\right)^2 - 2\frac{|z-x_r|}{\tau} \cos \theta + 1}} = \frac{\sin \beta}{\frac{|z-x_r|}{\tau}},$$

To estimate the last integral  $J$ , we consider

(a) For  $r < |z - x_r| < r + 1$ ,

$$J \leq \int_{r-\eta}^r \frac{(r+\tau)^{\frac{\alpha}{2}-1}}{(r-\tau)^{1-\frac{\alpha}{2}}} d\tau \leq C_{\alpha,r}.$$

(b) For  $|z - x_r| \geq r + 1$ , obviously,

$$J \sim \frac{1}{|z - x_r|^2}, \text{ for } |z - x_r| \text{ large.}$$

In summary,

$$I_{21}(x, z) \sim \begin{cases} 1, & \text{for } |z - x_r| \text{ near } r, \\ |z - x_r|^n, & \text{for } |z - x_r| \text{ large.} \end{cases}$$

Therefore, by (49), as  $k \rightarrow \infty$ ,

$$I_2 = \delta^\alpha \int_{|z-x_r|>r} \frac{u_k(z) - u(z)}{(|z - x_r|^2 - r^2)^{\alpha/2}} I_{21}(x, z) dz \rightarrow 0. \quad (60)$$

Now what remains is to estimate

$$I_1 = \delta^\alpha \int_{|z-x_r|>r} \frac{u_k(z) - u(z)}{(|z - x_r|^2 - r^2)^{\frac{\alpha}{2}}} I_{11}(x, z) dz,$$

where

$$I_{11}(x, z) = \int_{\substack{|y-x|>\delta \\ |y-x_r|<r-\eta}} \frac{(r^2 - |y - x_r|^2)^{\frac{\alpha}{2}} dy}{(|x - y|^2 - \delta^2)^{\frac{\alpha}{2}} |x - y|^n |z - y|^n}.$$

$$I_{11}(x, z) \leq \frac{r^\alpha}{\delta^n} \int_{\substack{|y-x|>\delta \\ |y-x_r|<r-\eta}} \frac{dy}{(|x - y|^2 - \delta^2)^{\frac{\alpha}{2}} |z - y|^n} \quad (61)$$

$$\leq \frac{r^\alpha}{\delta^n (|z - x_r| - r + \eta)^n} \int_{\delta < |y-x| < 2r} \frac{dy}{(|x - y|^2 - \delta^2)^{\frac{\alpha}{2}}} \quad (62)$$

$$= \frac{r^\alpha}{\delta^n (|z - x_r| - r + \eta)^n} \int_\delta^{2r} \frac{\omega_{n-1} \tau^{n-1} d\tau}{(\tau^2 - \delta^2)^{\frac{\alpha}{2}}} \quad (63)$$

$$\leq \frac{C}{|z - x_r|^n}. \quad (64)$$

By (49), as  $k \rightarrow \infty$ , we have  $I_1 \rightarrow 0$ . This verifies (56) and hence completes the proof.

**Lemma 3.2**

$$\lim_{r \rightarrow 0} \frac{1}{r^\alpha} [u(x) - \varepsilon_\alpha^{(r)} * u(x)] = c(-\Delta)^{\frac{\alpha}{2}} u(x). \quad (65)$$

where  $c = \frac{\Gamma(n/2)}{\pi^{\frac{n}{2}+1}} \sin \frac{\pi\alpha}{2}$ .

**Proof.**

$$\begin{aligned} & \frac{1}{r^\alpha} [u(x) - \varepsilon_\alpha^{(r)} * u(x)] \\ &= \frac{1}{r^\alpha} u(x) - c \int_{|y-x|>r} \frac{u(y)}{(|x-y|^2 - r^2)^{\frac{\alpha}{2}} |x-y|^n} dy \\ &= c \int_{|y-x|>r} \frac{u(x) - u(y)}{(|x-y|^2 - r^2)^{\frac{\alpha}{2}} |x-y|^n} dy. \end{aligned} \quad (66)$$

Here we have used the property that

$$\int_{|y-x|>r} \varepsilon_\alpha^{(r)}(x-y) = 1.$$

Compare (66) with

$$(-\Delta)^{\frac{\alpha}{2}} u(x) = \lim_{r \rightarrow 0} \int_{|y-x|>r} \frac{u(x) - u(y)}{|x-y|^{\alpha+n}} dy.$$

One may expect that

$$\lim_{r \rightarrow 0} \int_{|y-x|>r} \frac{u(x) - u(y)}{|x-y|^{\alpha+n}} dy = \lim_{r \rightarrow 0} \int_{|y-x|>r} \frac{u(x) - u(y)}{(|x-y|^2 - r^2)^{\frac{\alpha}{2}} |x-y|^n} dy.$$

Indeed, consider

$$\begin{aligned} & \int_{|y-x|>r} \frac{u(x) - u(y)}{|x-y|^n} \left( \frac{1}{(|x-y|^2 - r^2)^{\frac{\alpha}{2}}} - \frac{1}{|x-y|^\alpha} \right) dy \\ &= \int_{r < |y-x| < 1} \frac{u(x) - u(y)}{|x-y|^n} \left( \frac{1}{(|x-y|^2 - r^2)^{\frac{\alpha}{2}}} - \frac{1}{|x-y|^\alpha} \right) dy \\ & \quad + \int_{|y-x| \geq 1} \frac{u(x) - u(y)}{|x-y|^n} \left( \frac{1}{(|x-y|^2 - r^2)^{\frac{\alpha}{2}}} - \frac{1}{|x-y|^\alpha} \right) dy \\ &= I_1 + I_2. \end{aligned} \quad (67)$$

It is easy to see that as  $r \rightarrow 0$ ,  $I_2$  tends to zero. Actually, same conclusion is true for  $I_1$ .

$$I_1 = \int_{r < |y-x| < 1} \frac{\nabla u(x)(y-x) + O(|y-x|^2)}{|x-y|^n} \left( \frac{1}{(|x-y|^2 - r^2)^{\frac{\alpha}{2}}} - \frac{1}{|x-y|^\alpha} \right) dy \quad (68)$$

$$\leq C \int_{r < |y-x| < 1} \frac{|x-y|^2}{|x-y|^n} \left( \frac{1}{(|x-y|^2 - r^2)^{\frac{\alpha}{2}}} - \frac{1}{|x-y|^\alpha} \right) dy \quad (69)$$

$$= C \int_r^1 \frac{\tau^2}{\tau^n} \left( \frac{1}{(\tau^2 - r^2)^{\frac{\alpha}{2}}} - \frac{1}{\tau^\alpha} \right) \tau^{n-1} d\tau \quad (70)$$

$$\leq C \int_1^\infty \left( \frac{1}{r^\alpha (s^2 - 1)^{\frac{\alpha}{2}}} - \frac{1}{r^\alpha s^\alpha} \right) s r^2 ds \quad (71)$$

$$= C r^{2-\alpha} \int_1^\infty \left( \frac{s^\alpha - (s^2 - 1)^{\frac{\alpha}{2}}}{(s^2 - 1)^{\frac{\alpha}{2}} s^\alpha} \right) s ds. \quad (72)$$

Equation (68) follows from the Taylor expansion. Due to symmetry, we have

$$\int_{r < |y-x| < 1} \frac{\nabla u(x)(y-x)}{|x-y|^n} \left( \frac{1}{(|x-y|^2 - r^2)^{\frac{\alpha}{2}}} - \frac{1}{|x-y|^\alpha} \right) dy = 0$$

and get (69). By letting  $|y-x| = \tau$  and  $\tau = rs$  respectively, one obtains (70) and (71). It is easy to see that the integral in (72) converges near 1. To see that it also converges near infinity, we estimate

$$s^\alpha - (s^2 - 1)^{\frac{\alpha}{2}}.$$

Let  $f(t) = t^{\alpha/2}$ . By the *mean value theorem*,

$$\begin{aligned} f(s^2) - f(s^2 - 1) &= f'(\xi)(s^2 - (s^2 - 1)) \\ &= \frac{\alpha}{2} \xi^{\frac{\alpha}{2}-1} \sim s^{\alpha-2}, \text{ for } s \text{ sufficiently large.} \end{aligned}$$

This implies that

$$\frac{s^\alpha - (s^2 - 1)^{\frac{\alpha}{2}}}{(s^2 - 1)^{\frac{\alpha}{2}} s^\alpha} s \sim \frac{s^{\alpha-2} s}{(s^2 - 1)^{\frac{\alpha}{2}} s^\alpha} \sim \frac{1}{s^{1+\alpha}}.$$

Now it is obvious that (72) converges near infinity. Thus we have

$$\int_1^\infty \left( \frac{s^\alpha - (s^2 - 1)^{\frac{\alpha}{2}}}{(s^2 - 1)^{\frac{\alpha}{2}} s^\alpha} \right) s ds < \infty.$$

Since  $0 < \alpha < 2$ , as  $r \rightarrow 0$ , (72) goes to zero, i.e.  $I_1$  converges to zero. Together with (66) and (67), we get (65). This proves the lemma.

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